DERIVATION OF EQUATIONS OF MOTION FOR THE START OF AN ELECTRIC HOIST BY MEANS OF LAGRANGE EQUATIONS OF THE SECOND KIND

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Summary: Solution of the start of an electric hoist is generally solved as the system with 2 DOF; that is, translation motion of the hoist and rotational motion (swinging) of a load. However, in case of bulkier load can occur (e.g. due to wind forces) additional rotation on the wire rope fitting and may cause exceeding of allowable angle of the load deflection. This detrimental swinging can affect service life of wire ropes. It is necessary to receive better prerequisite for the solution of a swinging load.

Key words: Electric hoist, swinging of a load, DOF, Lagrange equations of the second kind

1. INTRODUCTION

Swinging of a load is usually harmful effect. This dynamic motion is induced by the inertia of the load as the crane (or electric hoist) powers up or brakes during trolleying or travel. Consider a traveling electric hoist – when braking occurs and electric hoist slows, the load will continue and pull forward on the hoist. After the hoist stops, the load will continue to swing, pendulum fashion. The period of swing will vary with hoist line length, the distance from load CG to the suspension point on the fitting. Thus, there are three generalized coordinates, i.e. three degrees of freedom (DOF) as will treated in next chapter.

2. START OF THE ELETRIC HOIST

2.1 Entry conditions

Consider the electric hoist (see fig. 1) with mass m_2 [kg] on which the wire rope with length l [m] is suspended. Assuming that wire rope is absolutely rigid and intangible. At the end of wire rope is fitted the load with mass m_4 [kg] and moment of inertia I_{4S} [kg.m^2] relative to the CG of load. Electric hoist is powered by gearmotor with starting torque M [N.m], its moment of inertia including rotating parts reduced to input shaft is I_M [kg.m^2]. Ratio between gear motor and traveling wheels is i [-]. Electric hoist has four traveling wheels and each wheel has a moment of inertia I_3 [kg.m^2] and radius r [m]. Distance from load CG to the suspension point on the fitting is r_S [m].

As stated in abstract of this article, this system has three DOF, i.e. three generalized coordinates. First generalized coordinate is x and entails path of electric hoist. Second

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generalized coordinate is φ – angle between plumb line and wire rope. Third generalized coordinate is ψ – angle between plumb line and swinging load.

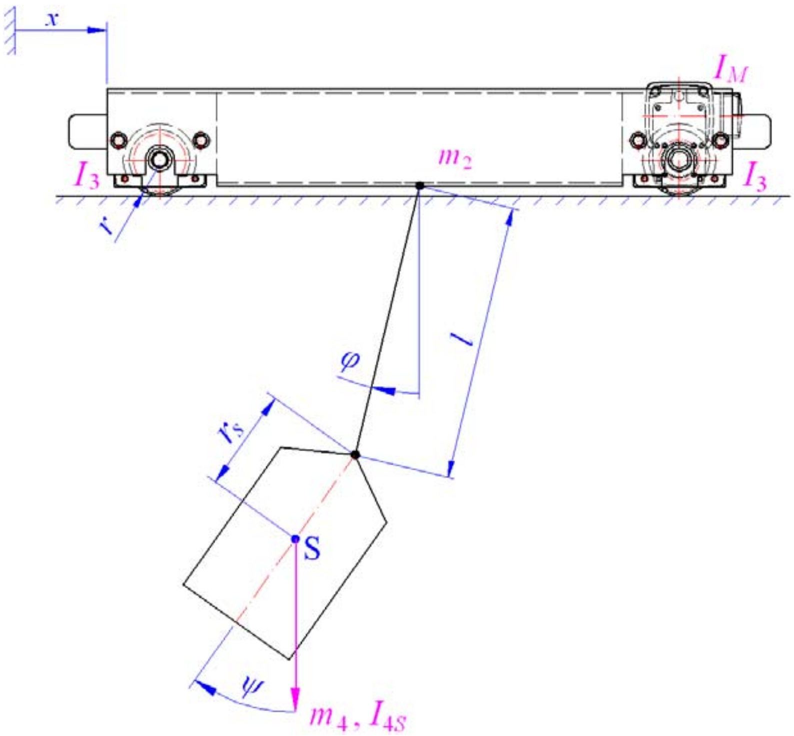

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Fig. 1 – Schematic of the electric hoist with swinging load

2.2 Lagrange equations of the second kind

System with three DOF has to be described by means of three independent equations. Every one equation poses one degree of freedom.

For this conservative system have Lagrange equations succeeding form:

$$\frac{d}{dt}\left(\frac{\partial W}{\partial \dot{x}}\right) - \frac{\partial W}{\partial x} = F = \frac{M \cdot i}{r} \tag{1}$$

$$\frac{d}{dt}\left(\frac{\partial W}{\partial \dot{\varphi}}\right) - \frac{\partial W}{\partial \varphi} = M = 0 \tag{2}$$

$$\frac{d}{dt}\left(\frac{\partial W}{\partial \dot{\psi}}\right) - \frac{\partial W}{\partial \psi} = M = 0 \tag{3}$$

where: W – Lagrangian, $W = W_k - W_p$ [J] (4)

W_k – kinetic energy of the system [J]

W_p – potential energy of the system [J]

2.2.1 Kinetic energy of the system

Kinetic energy of the system W_k consists of energy of translational motion and rotational energy. It's possible to describe velocity of the hoist as derivative path according to time and gets that $v = \dot{x}$ and angular velocity (in general) $\omega = \frac{v}{r} = \frac{\dot{x}}{r}$. Substituting we get

$$W_k = \frac{1}{2} \cdot m_2 \cdot \dot{x}^2 + \frac{1}{2} \cdot I_M \cdot i^2 \cdot \left(\frac{\dot{x}}{r}\right)^2 + 4 \cdot \frac{1}{2} \cdot I_3 \cdot \left(\frac{\dot{x}}{r}\right)^2 + \frac{1}{2} \cdot m_4 \cdot v_S^2 + \frac{1}{2} \cdot I_{4S} \cdot \dot{\psi}^2 \qquad (5)$$

Follow from equation (5), velocity of the load v_4 is unknown. For validity of this equation it's necessary to express this unknown as the function of generalized coordinates. Solving unknown velocity v_4 is possible through analysis of velocities (see the fig. 2).

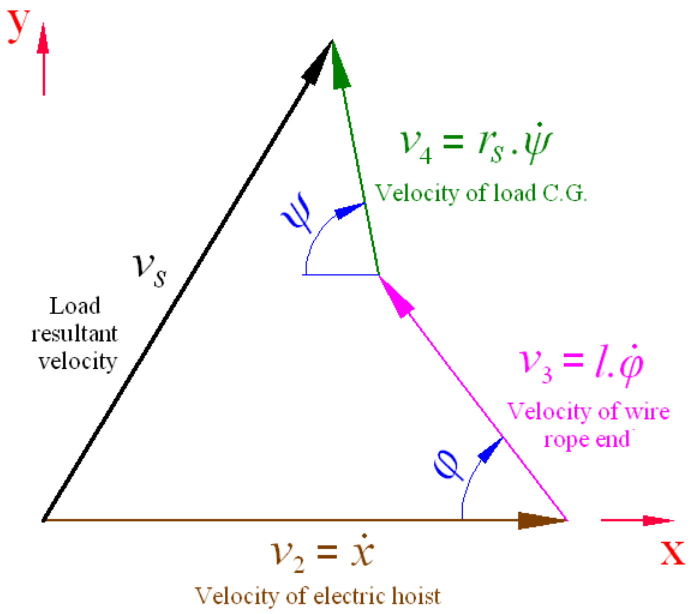

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Fig. 2 – Velocity analysis

According to fig. 2 we get components of velocity v_x and v_y:

$$v_{sx} = v_2 - v_3 \cdot \cos(\varphi) - v_4 \cdot \cos(\psi) = \dot{x} - l \cdot \dot{\varphi} \cdot \cos(\varphi) - r_S \cdot \dot{\psi} \cdot \cos(\psi) \qquad (6)$$

$$v_{sy} = v_3 \cdot \sin(\varphi) + v_4 \cdot \sin(\psi) = l \cdot \dot{\varphi} \cdot \sin(\varphi) + r_S \cdot \dot{\psi} \cdot \sin(\psi) \qquad (7)$$

Resultant velocity is then given by algebraic sum

$$\begin{aligned} v_s^2 = v_{sx}^2 + v_{sy}^2 &= \dot{x}^2 + l^2 \cdot \dot{\varphi}^2 \cdot \cos^2(\varphi) + r_S^2 \cdot \dot{\psi}^2 \cdot \cos^2(\psi) - 2 \cdot \dot{x} \cdot l \cdot \dot{\varphi} \cdot \cos(\varphi) - \\ &- 2 \cdot \dot{x} \cdot r_S \cdot \dot{\psi} \cdot \cos(\psi) + 2 \cdot l \cdot r_S \cdot \dot{\varphi} \cdot \dot{\psi} \cdot \cos(\varphi) \cdot \cos(\psi) + l \cdot \dot{\varphi}^2 \cdot \sin^2(\varphi) + \\ &+ 2 \cdot l \cdot r_S \cdot \dot{\varphi} \cdot \dot{\psi} \cdot \sin(\varphi) \cdot \sin(\psi) + r_S^2 \cdot \dot{\psi}^2 \cdot \sin^2(\psi) \end{aligned} \qquad (8)$$

Rearranging equation (8) we have

$$\begin{aligned} v_s^2 &= \dot{x}^2 + l^2 \cdot \dot{\varphi}^2 + r_S^2 \cdot \dot{\psi}^2 - 2 \cdot \dot{x} \cdot [l \cdot \dot{\varphi} \cdot \cos(\varphi) + r_S \cdot \dot{\psi} \cdot \cos(\psi)] + \\ &+ 2 \cdot l \cdot r_S \cdot \dot{\varphi} \cdot \dot{\psi} \cdot [\cos(\varphi) \cdot \cos(\psi) + \sin(\varphi) \cdot \sin(\psi)] \end{aligned}$$

Kinetic energy after substitution equation (8) to the equation (5)

$$W_k = \frac{1}{2}\cdot m_2\cdot \dot{x}^2 + \frac{1}{2}\cdot I_M\cdot i^2\cdot\left(\frac{\dot{x}}{r}\right)^2 + 4\cdot\frac{1}{2}\cdot I_3\cdot\left(\frac{\dot{x}}{r}\right)^2 + \frac{1}{2}\cdot I_{4S}\cdot\dot{\psi}^2 + \\ +\frac{1}{2}\cdot m_4\cdot\left\{\begin{matrix}\dot{x}^2 + l^2\cdot\dot{\varphi}^2 + r_S^2\cdot\dot{\psi}^2 - 2\cdot\dot{x}\cdot\left[l\cdot\dot{\varphi}\cdot\cos(\varphi) + r_S\cdot\dot{\psi}\cdot\cos(\psi)\right] + \\ + 2\cdot l\cdot r_S\cdot\dot{\varphi}\cdot\dot{\psi}\cdot\cos(\varphi-\psi)\end{matrix}\right\} \tag{9}$$

Rearranging equation (9) we get

$$W_k = \frac{1}{2}\cdot\left\{\begin{matrix} m_2\cdot\dot{x}^2 + I_M\cdot i^2\cdot\left(\frac{\dot{x}}{r}\right)^2 + 2\cdot I_3\cdot\left(\frac{\dot{x}}{r}\right)^2 + I_{4S}\cdot\dot{\psi}^2 + \\ + m_4\cdot\left\{\begin{matrix}\dot{x}^2 + l^2\cdot\dot{\varphi}^2 + r_S^2\cdot\dot{\psi}^2 - 2\cdot\dot{x}\cdot\left[l\cdot\dot{\varphi}\cdot\cos(\varphi) + r_S\cdot\dot{\psi}\cdot\cos(\psi)\right] + \\ + 2\cdot l\cdot r_S\cdot\dot{\varphi}\cdot\dot{\psi}\cdot\cos(\varphi-\psi)\end{matrix}\right\}\end{matrix}\right\} \tag{10}$$

2.2.2 Potential energy of the system

If we choose reference level of potential energy, where potential energy is zero as the track of the electric hoist, potential energy of the system is then given by

$$W_p = -m_4\cdot g\cdot\left[l\cdot\cos(\varphi) + r_S\cdot\cos(\psi)\right] \tag{11}$$

2.2.3 Lagrangian

The Lagrangian is a difference in kinetic and potential energy of the system, therefore

$$W = W_k - W_p = \frac{1}{2}\cdot\left\{\begin{matrix} m_2\cdot\dot{x}^2 + I_M\cdot i^2\cdot\left(\frac{\dot{x}}{r}\right)^2 + 4\cdot I_3\cdot\left(\frac{\dot{x}}{r}\right)^2 + I_{4S}\cdot\dot{\psi}^2 + \\ + m_4\cdot\left\{\begin{matrix}\dot{x}^2 + l^2\cdot\dot{\varphi}^2 + r_S^2\cdot\dot{\psi}^2 - 2\cdot\dot{x}\cdot\left[l\cdot\dot{\varphi}\cdot\cos(\varphi) + r_S\cdot\dot{\psi}\cdot\cos(\psi)\right] + \\ + 2\cdot l\cdot r_S\cdot\dot{\varphi}\cdot\dot{\psi}\cdot\cos(\varphi-\psi)\end{matrix}\right\}\end{matrix}\right\} + \\ + m_4\cdot g\cdot\left[l\cdot\cos(\varphi) + r_S\cdot\cos(\psi)\right] \tag{12}$$

2.2.4 Derivative of Lagrangian with respect to coordinates

In order to assemblage of Lagrange equations we have to determine partial derivatives with respect to appropriate generalized coordinate.

Differentiation with respect to *x*

$$\frac{\partial W}{\partial \dot{x}} = \frac{1}{2}\cdot\left[m_2\cdot 2\dot{x} + I_M\cdot i^2\cdot\frac{2\dot{x}}{r^2} + 4\cdot I_3\cdot\frac{2\dot{x}}{r^2} + m_4\cdot 2\dot{x} - m_4\cdot 2\cdot\left(l\cdot\dot{\varphi}\cdot\cos(\varphi) + r_S\cdot\dot{\psi}\cdot\cos(\psi)\right)\right]$$

$$\frac{\partial W}{\partial \dot{x}} = m_2\cdot\dot{x} + I_M\cdot i^2\cdot\frac{\dot{x}}{r^2} + 4\cdot I_3\cdot\frac{\dot{x}}{r^2} + m_4\cdot\dot{x} - m_4\cdot\left(l\cdot\dot{\varphi}\cdot\cos(\varphi) + r_S\cdot\dot{\psi}\cdot\cos(\psi)\right) \tag{13}$$

$$\frac{d}{dt}\left(\frac{\partial W}{\partial \dot{x}}\right) = m_2 \cdot \ddot{x} + I_M \cdot i^2 \cdot \frac{\ddot{x}}{r^2} + 4 \cdot I_3 \cdot \frac{\ddot{x}}{r^2} + m_4 \cdot \ddot{x} - m_4 \cdot l \cdot \ddot{\varphi} \cdot \cos(\varphi) + m_4 \cdot l \cdot \dot{\varphi} \cdot \sin(\varphi) \cdot \dot{\varphi} - \\ - m_4 \cdot r_S \cdot \ddot{\psi} \cdot \cos(\psi) + m_4 \cdot r_S \cdot \dot{\psi} \cdot \sin(\psi) \cdot \dot{\psi} \tag{14}$$

$$\frac{\partial W}{\partial x} = 0 \tag{15}$$

Differentiation with respect to φ

$$\frac{\partial W}{\partial \dot{\varphi}} = \frac{1}{2} \cdot m_4 \cdot \left[l^2 \cdot 2 \cdot \dot{\varphi} - 2 \cdot \dot{x} \cdot l \cdot \cos(\varphi) + 2 \cdot l \cdot r_S \cdot \dot{\psi} \cdot \cos(\varphi - \psi) \right] \tag{16}$$

$$\frac{\partial W}{\partial \dot{\varphi}} = m_4 \cdot l^2 \cdot \dot{\varphi} \; - \; m_4 \cdot \dot{x} \cdot l \cdot \cos(\varphi) \; + \; m_4 \cdot l \cdot r_S \cdot \dot{\psi} \cdot \cos(\varphi - \psi) \tag{17}$$

$$\frac{d}{dt}\left(\frac{\partial W}{\partial \dot{\varphi}}\right) = m_4 \cdot l^2 \cdot \ddot{\varphi} - m_4 \cdot \ddot{x} \cdot l \cdot \cos(\varphi) + m_4 \cdot \dot{x} \cdot l \cdot \sin(\varphi) \cdot \dot{\varphi} + m_4 \cdot l \cdot r_S \cdot \ddot{\psi} \cdot \cos(\varphi - \psi) \\ - m_4 \cdot l \cdot r_S \cdot \dot{\psi} \cdot \sin(\varphi - \psi) \cdot (\dot{\varphi} - \dot{\psi}) \tag{18}$$

$$\frac{\partial W}{\partial \varphi} = \frac{1}{2} \cdot m_4 \cdot 2 \cdot \dot{x} \cdot l \cdot \dot{\varphi} \cdot \sin(\varphi) - \frac{1}{2} \cdot m_4 \cdot 2 \cdot l \cdot r_S \cdot \dot{\varphi} \cdot \dot{\psi} \cdot \sin(\varphi - \psi) - m_4 \cdot g \cdot l \cdot \sin(\varphi) \tag{19}$$

$$\frac{\partial W}{\partial \varphi} = m_4 \cdot \dot{x} \cdot l \cdot \dot{\varphi} \cdot \sin(\varphi) - m_4 \cdot l \cdot r_S \cdot \dot{\varphi} \cdot \dot{\psi} \cdot \sin(\varphi - \psi) - m_4 \cdot g \cdot l \cdot \sin(\varphi) \tag{20}$$

Differentiation with respect to ψ

$$\frac{\partial W}{\partial \dot{\psi}} = \frac{1}{2} \cdot \begin{bmatrix} I_{4S} \cdot 2 \cdot \dot{\psi} + m_4 \cdot r_S^2 \cdot 2 \cdot \dot{\psi} - m_4 \cdot 2 \cdot \dot{x} \cdot r_S \cdot \cos(\psi) + \\ + m_4 \cdot 2 \cdot l \cdot r_S \cdot \dot{\varphi} \cdot \cos(\varphi - \psi) \end{bmatrix} \tag{21}$$

$$\frac{\partial W}{\partial \dot{\psi}} = I_{4S} \cdot \dot{\psi} + m_4 \cdot r_S^2 \cdot \dot{\psi} - m_4 \cdot \dot{x} \cdot r_S \cdot \cos(\psi) + m_4 \cdot l \cdot r_S \cdot \dot{\varphi} \cdot \cos(\varphi - \psi) \tag{22}$$

$$\frac{d}{dt}\left(\frac{\partial W}{\partial \dot{\psi}}\right) = I_{4S} \cdot \ddot{\psi} + m_4 \cdot r_S^2 \cdot \ddot{\psi} - m_4 \cdot \ddot{x} \cdot r_S \cdot \cos(\psi) + m_4 \cdot \dot{x} \cdot r_S \cdot \sin(\psi) \cdot \dot{\psi} + \\ + m_4 \cdot l \cdot r_S \cdot \ddot{\varphi} \cdot \cos(\varphi - \psi) - m_4 \cdot l \cdot r_S \cdot \dot{\varphi} \cdot \sin(\varphi - \psi) \cdot (\dot{\varphi} - \dot{\psi}) \tag{23}$$

$$\frac{\partial W}{\partial \psi} = \frac{1}{2} \cdot m_4 \cdot 2 \cdot \dot{x} \cdot r_S \cdot \dot{\psi} \cdot \sin(\psi) - \frac{1}{2} \cdot m_4 \cdot 2 \cdot l \cdot r_S \cdot \dot{\varphi} \cdot \dot{\psi} \cdot \sin(\varphi - \psi) - m_4 \cdot g \cdot r_S \cdot \sin(\psi) \tag{24}$$

$$\frac{\partial W}{\partial \psi} = m_4 \cdot \dot{x} \cdot r_S \cdot \dot{\psi} \cdot \sin(\psi) - m_4 \cdot l \cdot r_S \cdot \dot{\varphi} \cdot \dot{\psi} \cdot \sin(\varphi - \psi) - m_4 \cdot g \cdot r_S \cdot \sin(\psi) \tag{25}$$

2.2.5 Substitution of derivatives and determination of equations of motion

If all required derivatives are available, it is possible to determinate the equations of motion, referred to above in equations (1), (2) and (3).

After substituting derivatives we get

Equation of motion for generalized coordinate *x*

$$\ddot{x} \cdot \left(m_2 + I_M \cdot \frac{i^2}{r^2} + 4 \cdot \frac{I_3}{r^2} + m_4 \right) - m_4 \cdot l \cdot \ddot{\varphi} \cdot \cos(\varphi) + m_4 \cdot l \cdot \dot{\varphi}^2 \cdot \sin(\varphi) - m_4 \cdot r_S \cdot \ddot{\psi} \cdot \cos(\psi) + \\ + m_4 \cdot r_S \cdot \dot{\psi}^2 \cdot \sin(\psi) = \frac{M \cdot i}{r} \tag{26}$$

Equation of motion for generalized coordinate φ

$$m_4 \cdot l^2 \cdot \ddot{\varphi} + m_4 \cdot l \cdot r_S \cdot \ddot{\psi} \cdot \cos(\psi - \varphi) - m_4 \cdot l \cdot \ddot{x} \cdot \cos(\varphi) - m_4 \cdot l \cdot r_S \cdot \dot{\psi}^2 \cdot \sin(\psi - \varphi) + \\ + m_4 \cdot g \cdot l \cdot \sin(\varphi) = 0 \tag{27}$$

And finally equation of motion for generalized coordinate ψ

$$\ddot{\psi} \cdot \left(I_{4S} + m_4 \cdot r_S^2 \right) + m_4 \cdot l \cdot r_S \cdot \ddot{\varphi} \cdot \cos(\psi - \varphi) - m_4 \cdot r_S \cdot \ddot{x} \cdot \cos(\psi) + \\ + m_4 \cdot l \cdot r_S \cdot \dot{\varphi}^2 \cdot \sin(\psi - \varphi) + m_4 \cdot g \cdot r_S \cdot \sin(\psi) = 0 \tag{28}$$

It is apparent that obtained equations of motion pose system of nonlinear differential equations. This system can be solved either using numerical solution or by linearization equations on the assumption that angles φ and ψ have a small value. However, solution of these equations isn't aim of this article, hence it isn't here stated.

3. CONCLUSION

This system with 3 DOF represents an elaborate and reliable means for mathematical description of the start of the electric hoist with reference to swinging load and sequentially, this one may be used for operating and verification of the real electric hoist.

REFERENCES

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