OF $E_2/E_2/1/m$ QUEUEING SYSTEM SUBJECT TO OPERATE-DEPENDENT SERVER BREAKDOWNS

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Summary: The paper deals with modelling of a finite single-server queueing system with a server subject to breakdowns. Customers interarrival times and customers service times follow the Erlang distribution defined by the shape parameter k=2 and the scale parameter 2λ or 2μ respectively. We consider that server failures can occur when the server is busy (so called operate-dependent failures). Further we assume that service of a customer is interrupted by the occurrence of the server failure (the preemptive-repeat discipline. We assume that random variables relevant to server failures and repairs are exponentially distributed. The queueing system is modelled using method of stages. We present a state transition diagram, a system of linear equations describing the system behaviour in the steady-state and formulas for several performance measures computation. To validate the mathematical model a simulation model was created using simulation software Witness. At the end of the paper some graphical dependencies are shown.

Key words: $E_2/E_2/1/m$, queueing, method of stages, server breakdown, Witness

INTRODUCTION

Queueing systems represent a lot of practical systems we can find in technical practice, such as manufacturing, computer and telecommunication or transport systems. As we can see in many books devoted to the queueing theory, such as the books (1) or (2), in most common queueing models we often neglect the fact that a server is subject to failures. However in technical practice we often must not forget this fact because server failures can adversely affect performance measures of a studied queueing system. Therefore we are obliged to model the system as the unreliable queueing system in which the server is successively failure-free and broken.

We can mention two examples of queuing systems subject to server breakdowns in the field of transport. Let us consider a portal crane which loads/unloads containers on/from wagons in a container terminal. It is clear that the crane, which is the server in this case, can be broken from time to time. Or the second case that could be mentioned, authors of the paper (3) modelled a container unloader as a finite single-server queue with repairable server.

In the paper a modification of a queueing system which was presented in paper (4) is shown.

1. GENERAL ASSUMPTIONS AND NOTATION

Let us study a single server queueing system with a finite capacity equal to m, where $m>1$, that means there are in total m places for customers in the system – single place in the

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service and m-1 places intended for waiting of customers. Let us consider that customers are served one by one according to the FCFS service discipline.

Customers interarrival times follow the Erlang distribution with the shape parameter k=2 and the scale parameter 2λ; therefore the mean interarrival time is then equal to $\frac{2}{2\lambda}=\frac{1}{\lambda}$. Costumer service times are an Erlang random variable with the shape parameter k=2 as well, but with the scale parameter 2μ; thus the mean service time is equal to $\frac{2}{2\mu}=\frac{1}{\mu}$.

Let us assume that the server is successively failure-free (or available we can say) and broken. We assume that failures of the server can occur when the server is busy – we say that server failures are operate-dependent. Let us assume that times of overall server working until the breakdown occurrence are an exponential random variable with the parameter η; the mean time of overall server working until the breakdown occurrence is then equal to the reciprocal value of the parameter η. Times to repair are an exponential random variable as well, but with the parameter ξ; the mean time to repair is therefore equal to $\frac{1}{\xi}$.

As regards behaviour of customers at the moment of the failure, we will consider that the performed service of the customer under service is lost, the customer leaves the server and comes back to the queue if it is possible; otherwise it leaves the system and is rejected. We can call this discipline as preemptive-repeat.

To model our queueing system we can employ method of stages. The method exploits the fact that the Erlang distribution with the shape parameter k and the scale parameter denoted as $k\lambda$ or $k\mu$ is sum of k independent exponential distribution with the same parameter $k\lambda$ or $k\mu$. Therefore the queueing system can be modelled using Markov chains.

2. MATHEMATICAL MODEL OF THE QUEUEING SYSTEM

States of the system can be divided into two groups:

- The failure-free states are denoted by the notation k,v,o, where:
  - k represents the number of customers finding in the system, where $k \in \{0,1,...,m\}$,
  - v represents the terminated phase of customer arrival, where $v \in \{0,1\}$,
  - o represents the terminated phase of customer service, where $o \in \{0,1\}$.
- The states in which the server is broken are denoted by the notation Pk,v, where:
  - the letter P expresses failure of the server,
  - k represents the number of customers finding in the system, where $k \in \{0,1,...,m-1\}$,
  - v represents the terminated phase of customer arrival, where $v \in \{0,1\}$.

Let us illustrate a state transition diagram; the diagram is shown in Fig. 1. Vertices represent individual system states and oriented edges indicate possible transitions with the corresponding rate.

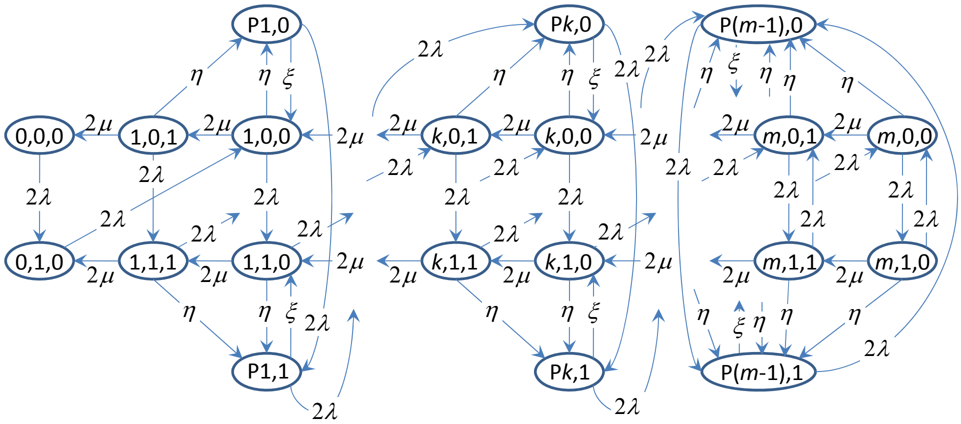

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Fig. 1 – The state transition diagram of the queueing model

The finite system of linear equations describing the behaviour of the system in the steady-state is:

$$2\lambda P_{0,0,0} = 2\mu P_{1,0,1}, \tag{1}$$

$$2\lambda P_{0,1,0} = 2\lambda P_{0,0,0} + 2\mu P_{1,1,1}, \tag{2}$$

$$(2\lambda + 2\mu + \eta)P_{1,0,1} = 2\mu P_{1,0,0}, \tag{3}$$

$$(2\lambda + 2\mu + \eta)P_{k,1,1} = 2\lambda P_{k,0,1} + 2\mu P_{k,1,0} \text{ for } k = 1,2,\ldots,m, \tag{4}$$

$$(2\lambda + 2\mu + \eta)P_{k,0,0} = 2\lambda P_{k-1,1,0} + 2\mu P_{k+1,0,1} + \xi P_{Pk,0} \text{ for } k = 1,2,\ldots,m-1, \tag{5}$$

$$(2\lambda + 2\mu + \eta)P_{k,1,0} = 2\lambda P_{k,0,0} + 2\mu P_{k+1,1,1} + \xi P_{Pk,1} \text{ for } k = 1,2,\ldots,m-1, \tag{6}$$

$$(2\lambda + 2\mu + \eta)P_{k,0,1} = 2\lambda P_{k-1,1,1} + 2\mu P_{k,0,0} \text{ for } k = 2,3,\ldots,m-1, \tag{7}$$

$$(2\lambda + 2\mu + \eta)P_{m,0,1} = 2\lambda P_{m-1,1,1} + 2\lambda P_{m,1,1} + 2\mu P_{m,0,0}, \tag{8}$$

$$(2\lambda + 2\mu + \eta)P_{m,0,0} = 2\lambda P_{m-1,1,0} + 2\lambda P_{m,1,0}, \tag{9}$$

$$(2\lambda + 2\mu + \eta)P_{m,1,0} = 2\lambda P_{m,0,0}, \tag{10}$$

$$(2\lambda + \xi)P_{P1,0} = \eta P_{1,0,1} + \eta P_{1,0,0}, \tag{11}$$

$$(2\lambda + \xi)P_{Pk,1} = \eta P_{k,1,1} + \eta P_{k,1,0} + 2\lambda P_{Pk,0} \text{ for } k = 1,2,\ldots,m-2, \tag{12}$$

$$(2\lambda + \xi)P_{Pk,0} = \eta P_{k,0,1} + \eta P_{k,0,0} + 2\lambda P_{P(k-1),1} \text{ for } k = 2,3\ldots,m-2, \tag{13}$$

$$(2\lambda + \xi)P_{P(m-1),0} = \eta P_{m-1,0,1} + \eta P_{m-1,0,0} + \eta P_{m,0,1} + \eta P_{m,0,0} + +2\lambda P_{P(m-2),1} + 2\lambda P_{P(m-1),1}, \tag{14}$$

$$(2\lambda + \xi)P_{P(m-1),1} = \eta P_{m-1,1,1} + \eta P_{m-1,1,0} + \eta P_{m,1,1} + \eta P_{m,1,0} + 2\lambda P_{P(m-1),0} \tag{15}$$

including normalisation equation:

$$P_{0,0,0} + P_{0,1,0} + \sum_{k=1}^{m}\sum_{v=0}^{1}\sum_{o=0}^{1} P_{k,v,o} + \sum_{k=1}^{m-1}\sum_{v=0}^{1} P_{Pk,v} = 1. \tag{16}$$

Please notice that equation (15) is linear combination of equations (1) up to (14), therefore we omit equation (15) and replace it by normalisation equation (16). Solving of linear equation system formed from equations (1) – (14), (16) we get stationary probabilities of the particular system states that are needed for computing of performance measures.

Let us consider three performance measures – the mean number of the customers in the service *ES*, the mean number of the waiting customers *EL* and the mean number of the broken servers *EP*. All of them can be computed according to the formula for the mean value of discrete random variable, where the random variable $S \in \{0,1\}$ is the number of costumers in the service, $L \in \{0, m-1\}$ the number of waiting customers and $P \in \{0,1\}$ the number of broken servers. For the mean number of the costumers in the service *ES* we can write:

$$ES = \sum_{k=1}^{m} \sum_{v=0}^{1} \sum_{o=0}^{1} P_{k,v,o} , \tag{17}$$

the mean number of the waiting costumers *EL* can be expressed by formula:

$$EL = \sum_{k=2}^{m} (k-1) \sum_{v=0}^{1} \sum_{o=0}^{1} P_{k,v,o} + \sum_{k=1}^{m-1} k \sum_{v=0}^{1} P_{Pk,v} , \tag{18}$$

and for the mean number of broken servers *EP* we get:

$$EP = \sum_{k=1}^{m-1} \sum_{v=0}^{1} P_{Pk,v} . \tag{19}$$

3. SIMULATION MODEL

To validate the mathematical model we created a simple simulation model using simulation software Witness 2008, which is intended for simulation of discrete event systems. The model is depicted in Fig. 2.

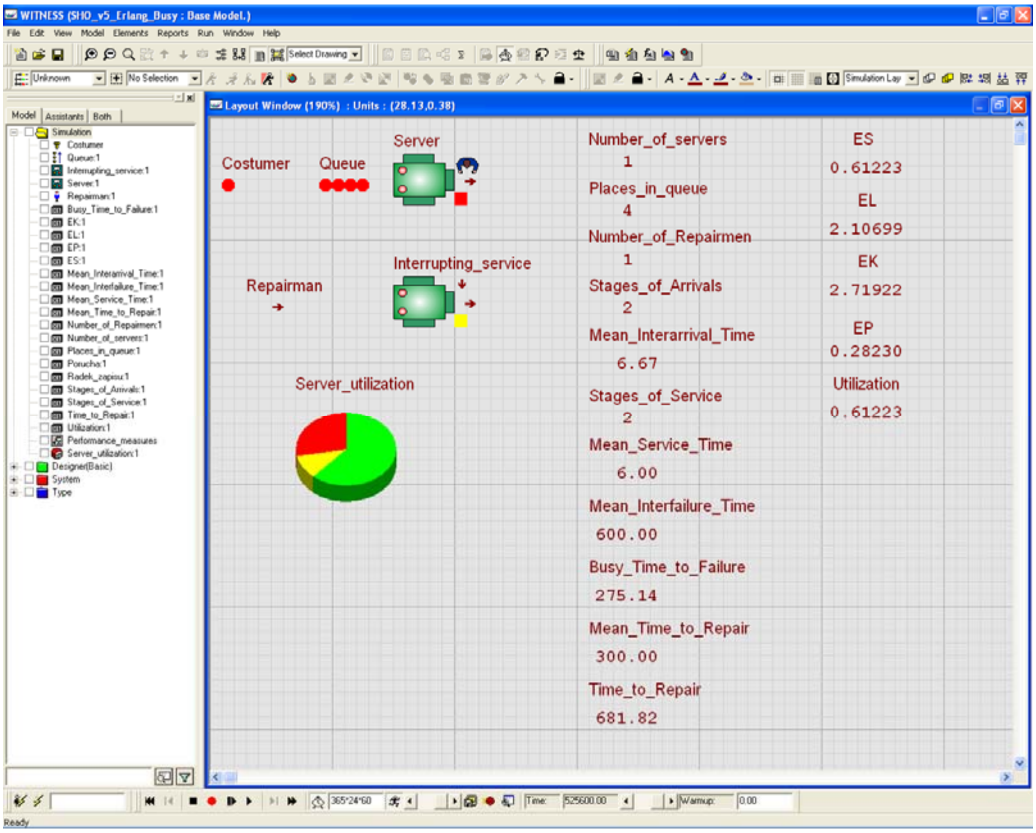

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Fig. 2 – The simulation model created in Witness 2008.

4. OUTCOMES OF EXECUTED EXPERIMENTS

Let us consider the studied queueing system with 5 places in the system. In Tab. 1 the values of applied random variables parameters are summarised.

Tab. 1 – Applied random variables parameters

Random variable (RV)	Applied parameters of RV
Interarrival times – Erlang RV	k=2; 2λ = 18 h^{-1}
Service times – Erlang RV	k=2; 2μ = 20 h^{-1}
Times of failure-free state – exponential RV	η=200^{-1}; 190^{-1};…, 20^{-1}; 10^{-1} h^{-1}
Times to repair – exponential RV	ξ=0.2 h^{-1}

Source: Author

For each value of the parameter η the stationary probabilities were computed numerically using software Matlab. On the basis of stationary probabilities knowledge we are able to compute the performance measures according to the corresponding formulas. Further we estimated the performance measures on the basis of simulation experiments, for each value of the parameter η we executed 30 simulation runs, each run was terminated after reaching simulation time equal to 525 600 minutes. On the basis of gained simulation outcomes we got 95% confidence intervals for the considered performance measures *ES*, *EL* and *EP* where T_l is the lower bound and T_u is the upper bound of the interval. Reached outcomes are summarized in Tab. 2. The graphical dependencies of individual performance measures on the reciprocal value of the parameter η are shown in Figs. 2, 3 and 4.

Tab. 2 – Summary of reached outcomes

$1/\eta$ [h]	Analytical results			Simulation results					
	ES	*EL*	*EP*	T_l for *ES*	T_u for *ES*	T_l for *EL*	T_u for *EL*	T_l for *EP*	T_u for *EP*
200	0,83012	1,36441	0,02075	0,83063	0,83359	1,35477	1,36498	0,01814	0,02179
190	0,82924	1,36744	0,02182	0,82976	0,83273	1,35895	1,36938	0,01925	0,02286
180	0,82825	1,37079	0,02301	0,82898	0,83184	1,36029	1,37191	0,02038	0,02399
170	0,82716	1,37453	0,02433	0,82784	0,83068	1,36479	1,37585	0,02171	0,02528
160	0,82593	1,37873	0,02581	0,82689	0,82989	1,36812	1,37916	0,02263	0,02640
150	0,82454	1,38347	0,02748	0,82538	0,82870	1,37290	1,38474	0,02413	0,02809
140	0,82296	1,38886	0,02939	0,82357	0,82742	1,37784	1,38949	0,02576	0,03014
130	0,82115	1,39506	0,03158	0,82157	0,82557	1,38115	1,39596	0,02783	0,03261
120	0,81904	1,40226	0,03413	0,81930	0,82352	1,38843	1,40232	0,03032	0,03521
110	0,81656	1,41072	0,03712	0,81723	0,82122	1,39675	1,40958	0,03276	0,03754
100	0,81361	1,42081	0,04068	0,81429	0,81863	1,40491	1,41968	0,03613	0,04128
90	0,81003	1,43303	0,04500	0,81083	0,81550	1,41426	1,43072	0,03967	0,04536
80	0,80560	1,44817	0,05035	0,80636	0,81125	1,42956	1,44721	0,04502	0,05084
70	0,79997	1,46738	0,05714	0,80128	0,80656	1,44611	1,46370	0,05046	0,05670
60	0,79259	1,49258	0,06605	0,79401	0,79936	1,47044	1,48931	0,05902	0,06551
50	0,78248	1,52709	0,07825	0,78432	0,79035	1,50659	1,52755	0,07057	0,07784
40	0,76779	1,57722	0,09597	0,76999	0,77642	1,55339	1,57530	0,08729	0,09502
30	0,74449	1,65669	0,12408	0,74791	0,75465	1,62362	1,64720	0,11293	0,12098
20	0,70188	1,80191	0,17547	0,70764	0,71393	1,76590	1,78653	0,16230	0,16983
10	0,59897	2,15184	0,29949	0,60763	0,61384	2,11030	2,13243	0,28298	0,29040

Source: Author

It can be seen from Tab. 2 that there are little differences between analytic and simulation outcomes; especially for lower values of the reciprocal value of parameter η we can see that some analytic outcomes lie outside the corresponding confidence intervals. But the differences are not essential and, in addition, can be partially explained by generally known disadvantages of simulation (for example using pseudo-random numbers). Therefore we can state that both models are valid.

As we can see in Fig. 3, increasing value of parameter η (or decreasing value of the reciprocal value of parameter η) causes decreasing of the mean number of customers in the system *ES*. This fact could be logically expected because more frequent failures mean lower fraction of time in which the server is able to serve incoming customers.

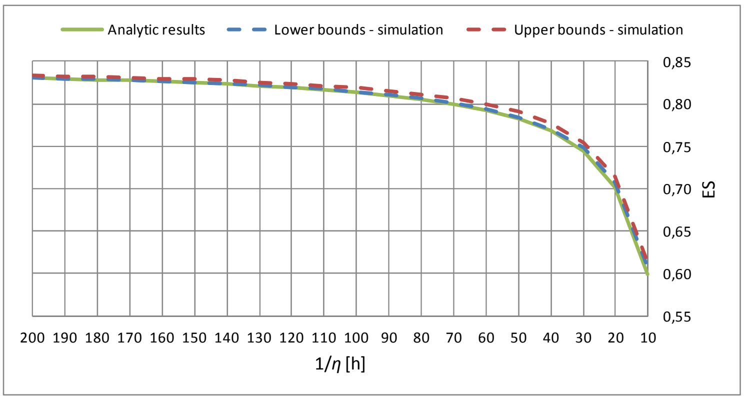

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Fig. 3 – The dependence of *ES* on parameter $1/\eta$

In Fig. 4 we can see that the mean number of waiting customers *EL* increases with decreasing reciprocal value of η because waiting of customers is prolonged due to more frequent failures.

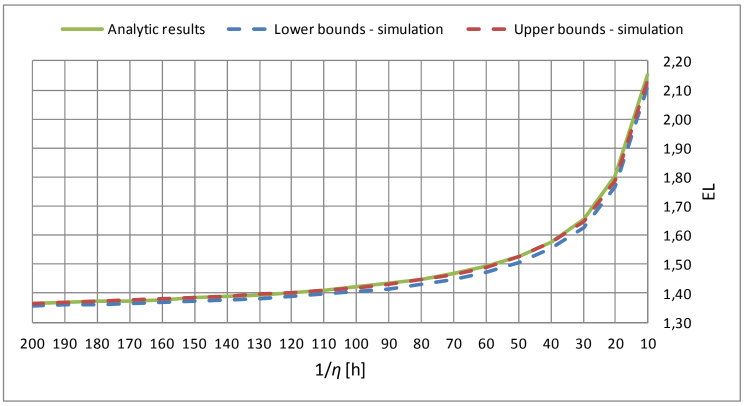

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Fig. 4 – The dependence of *EL* on parameter $1/\eta$

In Fig. 5 we can see that the dependency of the performance measure *EP* is increasing. This fact is obvious as well.

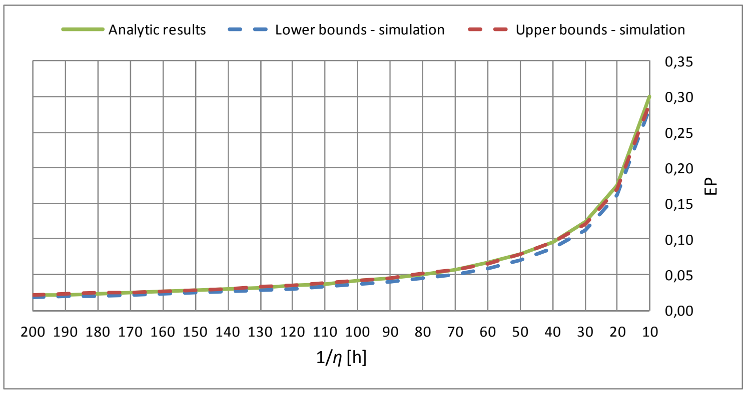

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Fig. 5 – The dependence of *EP* on parameter 1/η

5. CONCLUSIONS

In this paper we paid attention to the finite $E_2/E_2/1/m$ queue with the server subject to operate-dependent breakdowns. Further we considered that performed service of a customer operated at the moment of a server failure is lost and the customer either goes back to the queue or is rejected when the queue is full. We developed the state transition diagram and wrote the system of linear equations for the steady-state. The stationary probabilities can be numerically computed, for example, by using software Matlab. When we know the probabilities we are able to compute several performance measures we are interested in. Further we presented some numerical experiments executed with the model, to validate the mathematical model we created a simple simulation model using software Witness. We got some graphical dependencies of the selected performance measures on the reciprocal value of the parameter $1/\eta$.

In the future we would like to find the formula for the customer loss probability, because this performance measures is often very important for finite queueing systems. Further we would like to generalize the model for values of the shape parameter $k \geq 2$ and to program all necessary computations in Matlab.

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